Magnetohydrodynamic pipe flow

Part 2. High Hartmann number

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(Received 27 February 1962)

The paper presents an improved, second approximation for the laminar motion of a conducting liquid at high Hartmann number in non-conducting pipes of arbitrary cross-section under uniform transverse magnetic fields. A satisfactory comparison with the author's previously published experimental pressure gradient/flow results is made for the case of a circular pipe.

1. Introduction

The laminar flow of highly conducting liquids in pipes under uniform transverse magnetic fields was first examined by Hartmann (1937), although Williams (1930) had studied the flow of electrolytes under the same conditions somewhat earlier.

From the practical standpoint the problem is interesting because of the occurrence of pipe flow under transverse fields in magnetohydrodynamic pumps, generators, accelerators and flowmeters, while the academic interest arises because the problem is a linear one, reasonably easy to analyse and exhibiting one of the basic phenomena of magnetohydrodynamics, viz. the tendency of a magnetic field to suppress vorticity perpendicular to itself, in competition with viscosity tending to promote vorticity. The Hartmann number M or $Ba(\sigma/\eta)^{\frac{1}{2}}$ is well known to measure the extent to which the magnetic field prevails in this contest. The quantities B, a, σ and η are the imposed magnetic field, a typical pipe dimension, the fluid conductivity and viscosity respectively. When M is large, as it usually is in practice, the result of the contest is that the flow consists of a core devoid of vorticity perpendicular to the field surrounded by a viscous boundary layer in which the velocity falls rapidly to zero at the wall.

This paper treats the high-M case, which has the attraction that it is one of those problems in M.H.D. that are easier to solve than their counterparts in ordinary fluid mechanics. At high M the problem may be solved equally easily for pipes of *any* cross-section.

Chang & Lundgren (1961) did this for pipes with thin conducting walls, generalizing the author's (Shercliff 1956) asymptotic solution by removing an unnecessary restriction on the magnitude of the wall conductivity. Braginskii (1959) had considered the case of perfectly conducting walls. Sakao (1962) has deduced good approximate solutions by using a variational principle due to Tani in the cases of perfectly- and non-conducting walls. In this paper a second approximation to the solution is developed in terms of corrections of order M^{-1} in the case of non-conducting walls. This leads to an agreeably good comparison with the author's (1956) experimental results for the flow of mercury through a circular pipe under a transverse field.

2. The second approximation

The first approximation (Shercliff 1953, 1956) involved two assumptions, which are corrected in the second approximation: (*a*) viscous forces are negligible in the core, and (*b*) the velocity defect in the boundary layer is negligible in the evaluation of the flow rate Q and mean velocity v_0. Both (*a*) and (*b*) give rise to errors of order M^{-1}. We shall consider (*b*) first.

The boundary layers

The defect of volumetric flow rate per unit perimeter of wall is

$$\int_0^\infty (v_c - v)\, dn,$$

where v is the axial fluid velocity, v_c its local core value and n is distance along the inward normal (see figure 1). The notation is the same as in the 1956 paper except

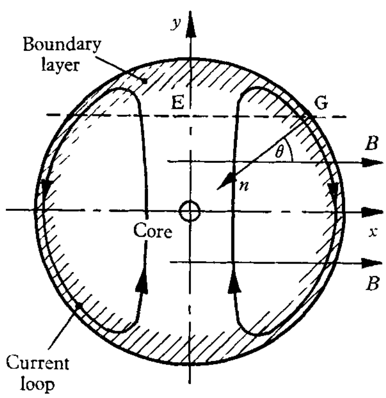

FIGURE 1. Cross-section of pipe.

that the variables are not non-dimensionalized. In evaluating this correction it is adequate to use the first approximation value

$$v = v_c\{1 - \exp(-Mn\cos\theta/a)\},$$

which is equation (15) of the 1956 paper. The angle θ is the inclination of the normal to the uniform transverse field. There are small regions where θ approaches $\frac{1}{2}\pi$ and where the boundary-layer theory fails. The flow defect per unit perimeter is seen to be $av_c/M\cos\theta$. The first approximation for v_c (equation (24) of the 1953 paper) may be used in this expression when the total flow defect is deduced by integration around the periphery.

For a non-conducting circular pipe of radius a, the first approximation for v_c is $(-\partial p/\partial z)\, a\cos\theta/B(\sigma\eta)^{\frac{1}{2}}$, which is equation (17) of the 1956 paper with $c = 0$.

The axial pressure gradient $(-\partial p/\partial z)$ is uniform over the cross-section. Here the flow defect per unit perimeter turns out to be constant and the total flow defect is simply $2\pi a^2(-\partial p/\partial z)/B^2\sigma$.

The core

As in the first approximation, we continue to neglect vorticity in the core transverse to the imposed magnetic field, so that v_c is independent of x, the co-ordinate parallel to the imposed field (see figure 1). That this gives rise to errors of lesser order will be verified later. We do, however, include the effect of viscous stresses associated with variation of v_c with y. The balance of axial forces requires that

$$Bj_{yc} = -\partial p/\partial z + \eta\, d^2 v_c/dy^2, \quad \text{a function of } y \text{ only.} \tag{1}$$

Here j_y is the induced current density in the y-direction, j_{yc} its value in the core. Over most of the core the last term is of order M^{-1} compared with the others and represents the correction to the first approximation, where it was neglected.

The balance of current flow across the line EG, of length $f(y)$, in figure 1 demands that

$$f(y)\, j_{yc} + \int (j_y - j_{yc})\, dx \text{ (across boundary layer at constant } y) = 0, \tag{2}$$

since $j_y = j_{yc}$ in the core and there is no current flow in the walls or across the y-axis if we assume symmetry of the cross-section. Since the boundary-layer thickness is of order $f(y)/M$ we can with sufficient accuracy replace j_{yc} by $(-\partial p/\partial z)/B$ in the integral in (2), so that

$$f(y)\, j_{yc} + \int \{j_y - (-\partial p/\partial z)/B\}\, dx = 0. \tag{3}$$

The integral proves to be simply related to the core velocity. In fact the first approximation here needs no correction. This is most easily seen if we consider the compound variable

$$u = v + \{H + (-\partial p/\partial z)\, x/B\}/(\sigma\eta)^{\frac{1}{2}},$$

in which H is the axial induced field, equal to

$$-\int_0^x j_y\, dx$$

for pipe cross-sections that are symmetrical in the y-axis. Here we are using rationalized M.K.S. units. From equations (7) and (8) of the 1953 paper it follows that

$$\partial u/\partial x + (a/M)\nabla^2 u = 0. \tag{4}$$

The solution to this equation has no boundary layer at the wall where x is positive, as has been pointed out by Wasow (1944) and Levinson (1950). The first approximation is to take $\partial u/\partial x = 0$ and $u = F(y)$ there, M being large. In fact

$$F(y) = (-\partial p/\partial z) f(y)/B(\sigma\eta)^{\frac{1}{2}}, \tag{5}$$

since v and H vanish at the wall $x = f(y)$. Evidently $\partial^2 u/\partial x^2$ is negligible in comparison with $\partial^2 u/\partial y^2$, which is of order u/a^2, provided $f(y)$ is not constant and the pipe not rectangular. Then (4) shows that $\partial u/\partial x$ is of order M^{-1} and that the change in u across the thin boundary layer at constant y is of order M^{-2} and is therefore negligible in the second approximation. It follows that

$$v_c + (\sigma\eta)^{-\frac{1}{2}} \int \{j_y - (-\partial p/\partial z)/B\}\, dx = 0, \tag{6}$$

just as in the first approximation. Combining (1), (3) and (6) gives

$$v_c = f(y)\,(-\partial p/\partial z + \eta\, d^2v_c/dy^2)/B(\sigma\eta)^{\frac{1}{2}}, \tag{7}$$

a differential equation for v_c. We are not so much interested in the detailed variation of v_c as in the volumetric flow integral $2\int v_c f(y)\, dy$, taken over the *whole* cross-section since the boundary-layer defect has already been evaluated. Inserting (7) directly into the integral gives the first approximation

$$2\int \{f(y)\}^2\, dy(-\partial p/\partial z)/B(\sigma\eta)^{\frac{1}{2}}$$

plus the viscous correction term

$$\frac{2a}{M}\int \{f(y)\}^2 \frac{d^2v_c}{dy^2}\, dy,$$

which may be integrated twice by parts to give

$$-\frac{2a}{M}\int v_c \frac{d^2}{dy^2}\{f(y)\}^2\, dy, \tag{8}$$

because dv_c/dy must be finite and $f(y)$ and v_c vanish at the extremes of y. Also $d\{f(y)\}^2/dy$ will be finite there unless the curvature of the wall is zero. In the correction term (8) it is adequate to use the first approximation for v_c.

If we specialize to the case of a circular pipe, $f(y) = (a^2-y^2)^{\frac{1}{2}}$, and (8) becomes

$$-\frac{8a}{M}\int_0^a v_c dy = -2\pi a^2(-\partial p/\partial z)/B^2\sigma,$$

which, by a coincidence, is equal to the boundary-layer defect.

The final expression for the flow rate Q, incorporating both corrections, is

$$Q = \frac{8a^3(-\partial p/\partial z)}{3B(\sigma\eta)^{\frac{1}{2}}} - \frac{4\pi a^2(-\partial p/\partial z)}{B^2\sigma},$$

and the resultant expression for the mean velocity is

$$v_0 = \frac{Q}{\pi a^2} = \frac{8a(-\partial p/\partial z)}{3\pi B(\sigma\eta)^{\frac{1}{2}}}\left(1 - \frac{3\pi}{2M}\right). \tag{9}$$

Figure 2 is a reproduction of the author's (1956) experimental results to which the plot of (9) has been added. Agreement is well within the errors of the original measurements, not only for the most accurate ones where $M = 121$ but also even when M is rather too small for (9) to be good enough. The agreement at low M is evidently fortuitous because of the failure of the graph to connect smoothly with the chain-dotted curve that represents the exact Bessel-function solution (Uflyand 1960; Uhlenbusch & Fischer 1961; Gold 1961) evaluated by Gold in the preceding paper. Note that the vertical scale is large and exaggerates the discrepancy, however.

The core flow correction may be arrived at in other ways. One is to take equation (4) and solve it iteratively in the form $\partial u/\partial x = -(a/M)\nabla^2 u_1$, where u_1 is the first approximation (5). Alternatively, the first approximation to v_c may be

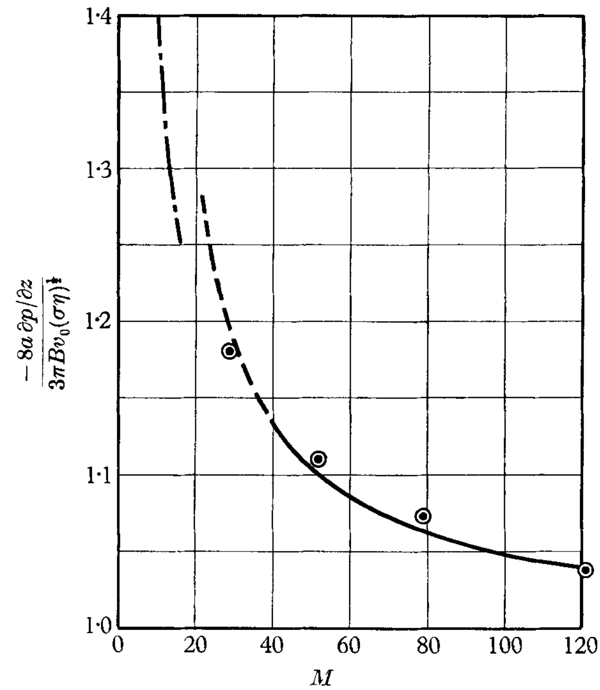

FIGURE 2. Non-dimensional (pressure gradient/flow) plotted against Hartmann number for circular pipes. ⊙, Experimental points (Shercliff 1956); ——, graph of $1/(1-3\pi/2M)$; —·—, exact Bessel-function solution.

inserted directly into the right-hand side of (7). Equally well the exact Bessel-function solution can be approximated at high M to give the same expression for v_c, as Gold (1962) has shown. These three methods have the slightly embarrassing feature that the resultant expression,

$$v_c = (-\partial p/\partial z)\{(a^2-y^2)^{\frac{1}{2}} - a^3/M(a^2-y^2)\}/B(\sigma\eta)^{\frac{1}{2}},$$

for a circular pipe, diverges negatively as $y \to a$, although the flow integral does not because the contributions from the obscure regions near $y = \pm a$ are so small. For this reason, and because it is more physical, the method given in full above has been preferred.

Finally we must check the initial assertion that v_c may still be taken as a function of y only. This may be done analytically from equation (4) but we shall adopt an equivalent but physically more instructive approach.

If V denotes electric potential, Ohm's Law has the components

$$j_x/\sigma = -\partial V/\partial x \quad \text{and} \quad j_y/\sigma = -\partial V/\partial y + Bv, \tag{10}$$

while continuity of current flow requires

$$\partial j_x/\partial x + \partial j_y/\partial y = 0.$$

In the second approximation the viscous forces cause j_{yc} to vary with y, and this leads to currents j_x and variations of V and $\partial V/\partial y$ in the x-direction. Then (10) indicates variations of v_c with x, since the effect of j_{yc} variation with x turns out to be smaller by two orders in M. In other words, the viscous stresses in the core associated with vorticity parallel to the field also promote some vorticity across the field by electrical action. In mathematical terms we have from (1), with the $\eta\,\partial^2 v/\partial x^2$ term still neglected, that

$$\frac{\partial j_{yc}}{\partial y} = \frac{\eta}{B}\frac{\partial^3 v_c}{\partial y^3} = -\frac{\partial j_x}{\partial x} = \sigma\frac{\partial^2 V}{\partial x^2}.$$

Thus
$$\frac{\partial^2 v_c}{\partial x^2} = \frac{\eta}{B^2\sigma}\frac{\partial^4 v_c}{\partial y^4} + \frac{1}{\sigma B}\frac{\partial^2 j_{yc}}{\partial x^2}. \tag{11}$$

The last term is seen from (1) to be

$$\frac{\eta}{B^2\sigma}\frac{\partial^2}{\partial y^2}\left(\frac{\partial^2 v_c}{\partial x^2}\right),$$

which is of order M^{-2} compared with the right-hand side of (11). It follows from (11) that x-wise variation of v_c is of order M^{-2} and may indeed be neglected in the second approximation.

To obtain a third, or better, approximation to the solution at high M it would be necessary to scrutinize the obscure regions near $y = \pm a$.

The author is grateful to Dr R. Gold for reviving his interest in the problem and for pointing out the asymptotic form of the Bessel-function solution.

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